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CONSTRAINT CLASSIFICATION ON THE UNIT
HYPERCUBE

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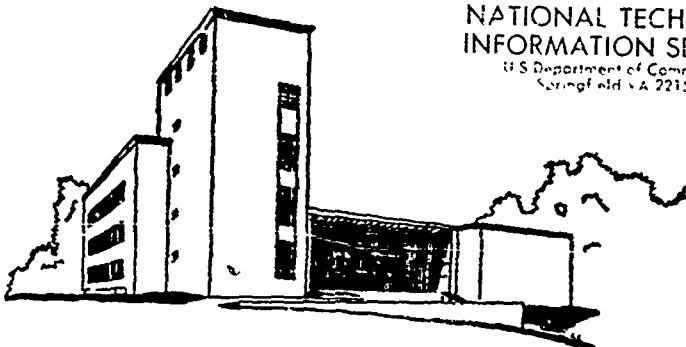
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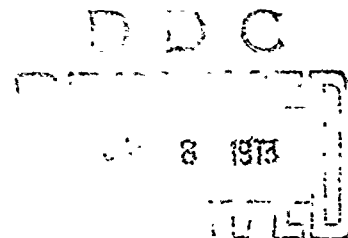
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This paper shows that there is a finite set of equivalence classes for constraints in the 0-1 programming problem. These equivalence classes have the property that exactly the same set of solutions are feasible for all constraints in the equivalence class. It is shown that these classes are determined by the relationship of the constraint to the dual of the hypercube. A function that indicates feasibility of 0-1 points is defined and is shown to be monotone over a vector partial ordering associated with the constraint and paired vertices of the hypercube. This allows for determination of equivalence classes based on identifying where the indicator function changes value. A search algorithm is presented for classifying the constraints and a method is presented for determining a "best" constraint from the class.

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Abstract

This paper shows that there is a finite set of equivalence classes for constraints in the 0-1 programming problem. These equivalence classes have the property that exactly the same set of solutions are feasible for all constraints in the equivalence class. It is shown that these classes are determined by the relationship of the constraint to the dual of the hypercube. A function that indicates feasibility of 0-1 points is defined and is shown to be monotone over a vector partial ordering associated with the constraint and paired vertices of the hypercube. This allows for determination of equivalence classes based on identifying where the indicator function changes value. A search algorithm is presented for classifying the constraints and a method is presented for determining a "best" constraint from the class.

I. Preliminaries

Let us consider two constraints $\sum a_i x_i' \geq b''$ and $\sum a_i' x_i' \geq b'$ where $x_i' = 0, 1 \quad \forall i$. It is obvious that while $a = (a_1, \dots, a_n)$ and $a' = (a_1', \dots, a_n')$ may be outwardly quite different, they may be redundant in the sense that the feasible sets for both constraints may be the same. This then raises the questions of how do we recognize these constraints, how many non-redundant constraints exist and how does one construct or choose "best" constraints from a redundant set. Balas and Jeroslow [4] have approached the first question by the use of canonical constraints. The question of choosing "best" constraints has been discussed Kianfar [7] and to some degree by Bowman and Nemhauser [6]. This paper addresses itself to all these questions and provides a means of classifying constraints into equivalence classes based on their feasible sets and then choosing a "best" constraint from these sets. In order to facilitate discussion we shall work with variables, x , that have been translated by $1/2 e$, $e = (1, \dots, 1)'$ from the x' space i.e. $x' = x + 1/2 e$. This translates 0, 1 values of x' into $-1/2, +1/2$ values of x . Consequently, the origin in the x space is the center of the hypercube whose vertices are the 0, 1 solutions desired. We also note that any constraint $\sum a_i x_i' \geq b'' - 1/2 \sum a_i$. In succeeding discussions we will be working only in the x space and thus right hand side values of the constraints will be for those after the translation into the x space, i.e., $b = b'' - 1/2 \sum a_i$.

Let K denote the unit hypercube, B the minimum ball containing K , and C the minimum octahedron containing K , i. e.,

$$K = [x \mid -1/2 e \leq x \leq +1/2 e],$$

$$B = [x \mid \sum_{i=1}^n x_i^2 \leq \frac{n}{4}],$$

$$C = [x \mid \sum_{i=1}^n |x_i| \leq \frac{n}{2}].$$

For a detailed discussion of these sets in relationship to integer programming, see [1,2,3].

If we denote by $bd(C)$ the boundary of the set C , the vertices of the hypercube can be described as 1) $bd(K) \cap bd(B)$, 2) $bd(K) \cap bd(C)$ or 3) $bd(B) \cap bd(C)$.

We note that the octahedron H is the intersection of 2^n half-spaces

$$\sum \delta_i x_i \leq n/2 \quad \delta_i = \pm 1 \quad (1)$$

and that these defining hyperplanes are tangent to B at the vertices of K .

Now consider a hyperplane $ax = b$ intersecting B . If we now construct all tangent planes to B at the intersection $\sum a_i x_i = b$ we form a displaced cone with vertex v at the intersection of the normal to $\sum a_i x_i = b$ and the tangent hyperplanes i. e., $v = \bar{\lambda} a$ where $\bar{\lambda} = \frac{n}{4b}$.

In projective geometry, v is the pole of the polar set $ax \leq b$ with respect to the conic B . We shall therefore refer to v as a pole. When referring to the pole of a particular constraint $ax \geq b$ we shall use the notation v is the pole of (a,b) . For other uses of poles and polar sets in integer programming see [2]. It is the pole v and its relationship to the extended facets of C that will determine the equivalence class of a constraint.

We note that if $b = 0$ then $\lambda = \infty$ and the tangent planes form a hypercylinder (i.e., have pole at infinity). This occurs only when $ax = b$ bisects B , that is, passes through the origin, $x=0$. When $b \neq 0$ the pole v may lie in either the feasible closed halfspace $ax \geq b$ called H^+ or the infeasible open halfspace $ax < b$ called H^- . One may consider the problem of choosing a "best" constraint as "pushing" the constraint away from v when $v \in H^-$ or "pulling" the constraint towards v when $v \in H^+$.

2. Equivalence Classes

Let us now consider the 2^n hyperplanes (1) that are the extended facets of C . We note that these facets occur in parallel pairs, i.e.,

$$\sum \delta_i x_i \leq n/2 \quad (2a)$$

$$\text{and} \quad \sum (-\delta_i) x_i \leq n/2 \quad (2b)$$

are parallel. There are obviously 2^{n-1} of these parallel pairs. Every set of the parallel pairs partition R^n into 3 sectors i.e., feasible to (2a) but not (2b), feasible to (2a) and (2b) and feasible to (2b) but not (2a).

The relationship of the pole of a constraint to the parallel pairs (2a) and (2b) will be fundamental to the establishment of equivalence classes on the constraints. In order to discuss pairs we shall need a unique designator for each pair. Let $\bar{\delta}$ be the unique designator of the parallel pair $\bar{\delta}$ and $-\bar{\delta}$. For uniqueness $\bar{\delta} = (\bar{\delta}_1, \bar{\delta}_2, \dots, \bar{\delta}_n)$ has the property $\sum \bar{\delta}_i \geq 0$ and if $\sum \bar{\delta}_i = 0$ then $\bar{\delta}_1 = -1$. We now wish to establish an indicator that relates the feasibility of the pole v of (a, b) to the constraints (2a) and (2b). This function will be defined over the 2^{n-1} parallel pairs denoted by $\bar{\delta}$.

Let $\Gamma(v, \bar{\delta})$ be a function on parallel pairs and v the pole of (a, b) that exhibits the following properties:

$$\Gamma(v, \bar{\delta}) = \begin{cases} 1 & \text{if } \sum \bar{\delta}_i v_i > \frac{n}{2} \\ 1 & \text{if } \sum \bar{\delta}_i v_i = \frac{n}{2} \text{ and } v \in H^+ \\ 0 & \text{if } \sum \bar{\delta}_i v_i = \frac{n}{2} \text{ and } v \in H^- \\ 0 & \text{if } -\frac{n}{2} < \sum \bar{\delta}_i v_i < \frac{n}{2} \\ 0 & \text{if } \sum \bar{\delta}_i v_i = -\frac{n}{2} \text{ and } v \in H^- \\ -1 & \text{if } \sum \bar{\delta}_i v_i = -\frac{n}{2} \text{ and } v \in H^+ \\ -1 & \text{if } \sum \bar{\delta}_i v_i < -\frac{n}{2} \end{cases}$$

$\Gamma(v, \bar{\delta})$ describes in which of the three partitions the point v lies. We thus have the following Lemma.

Lemma 1: The extended facets of the octahedron C partition R^n into

$3^{2^{n-1}}$ segments.

Proof: There are 2^{n-1} parallel pairs $\bar{\delta}$ and any point v is mapped into one of the three possible values under $\Gamma(v, \bar{\delta})$ depending on where v lies in relation to the partition of R^n by the pair $\bar{\delta}$. There are 2^{n-1} such mappings for a given v .

We now investigate the relationship of the pole v of the hyperplane $ax = b$, to feasible vertices of K in the halfspace $ax \geq b$.

Since the pole v is said to have $\lambda = \infty$ for $b = 0$, we extend our definition of Γ to include these values.

$$\Gamma(\infty, \bar{\delta}) = \begin{cases} 1 & \text{if } a\bar{\delta} < 0 \\ 0 & \text{if } a\bar{\delta} = 0 \\ -1 & \text{if } a\bar{\delta} > 0. \end{cases}$$

Recall that $1/2\bar{\delta}$ and $-1/2\bar{\delta}$ are vertices of K , i. e., solutions to our problem. The following Lemma relates the feasibility of these vertices to the function $\Gamma(v, \bar{\delta})$.

Lemma 2: Let $ax \geq b$ have a pole v then

if $b > 0$

a) $1/2\bar{\delta}$ is feasible if and only if $\Gamma(v, \bar{\delta}) = +1$

b) $-1/2\bar{\delta}$ is feasible if and only if $\Gamma(v, \bar{\delta}) = -1$

if $b < 0$

c) $1/2\bar{\delta}$ is infeasible if and only if $\Gamma(v, \bar{\delta}) = 1$

d) $-1/2\bar{\delta}$ is infeasible if and only if $\Gamma(v, \bar{\delta}) = -1$

and if $b = 0$

e) $1/2\bar{\delta}$ is infeasible if and only if $\Gamma(\infty, \bar{\delta}) = 1$ i. e., $a\bar{\delta} < 0$

f) $-1/2\bar{\delta}$ is infeasible if and only if $\Gamma(\infty, \bar{\delta}) = -1$ i. e., $a\bar{\delta} > 0$

Proof:

a) Assume $1/2 \bar{\delta}$ is feasible, then $1/2 \bar{\delta} a \geq b$ or multiplying by $\frac{n}{2b}$ we have $\frac{n}{4b} a \bar{\delta} \geq \frac{n}{2}$ or $v \bar{\delta} \geq \frac{n}{2}$. Since $b > 0$ then $v \in H^+$ and thus $\Gamma(v, \bar{\delta}) = 1$. Now let $\Gamma(v, \bar{\delta}) = 1$ then $v \bar{\delta} \geq \frac{n}{2}$ or $\frac{n}{4b} a \bar{\delta} \geq \frac{n}{2}$ or $1/2 a \bar{\delta} \geq b$. The proof of (b) is similar.

To prove (c) let $1/2 \bar{\delta}$ be infeasible then $a(1/2 \bar{\delta}) < b$ or multiplying by $\frac{n}{2b} < 0$ we get $(\frac{n}{4b}) a \bar{\delta} > \frac{n}{2}$ and thus $\Gamma(v, \bar{\delta}) = 1$. Now assume that $\Gamma(v, \bar{\delta}) = 1$. Since $b < 0$ then $v \in H^-$ and we have $(\frac{n}{4b}) a \bar{\delta} > \frac{n}{2}$ and consequently $1/2 a \bar{\delta} < b$. Again the proof of (d) is similar to that of (c). To prove (e) assume $1/2 \bar{\delta}$ is infeasible then $a(1/2) \bar{\delta} < 0$ or $a \bar{\delta} < 0$. Likewise if $a \bar{\delta} < 0$ then $1/2 a \bar{\delta} < 0$. Again the proof of (f) is similar to that of (e).

We have thus shown that we can use the location of the pole of (a, b) to yield information about feasible and infeasible vertices of K with respect to $ax \geq b$.

Definition. We say that two constraints (a, b) and (a', b') are in the same equivalence class E , if the set of feasible vertices of K are the same for both (a, b) and (a', b') .

We denote (a, b) in equivalence class E by $(a, b) \in E$.

Theorem 1. There are $(2 \cdot 3^{2^{n-1}} - 2^{2^{n-1}})$ equivalence classes for constraints in an n -dimensional hypercube.

Proof: From Lemma 2 we can classify opposite points $1/2 \bar{\delta}$ and $-1/2 \bar{\delta}$ by the function Γ . Thus

- a) $1/2 \bar{\delta}$ and $-1/2 \bar{\delta}$ are both feasible if and only if $b \leq 0$
and $\Gamma(v, \bar{\delta}) = 0$
- b) $1/2 \bar{\delta}$ is feasible and $-1/2 \bar{\delta}$ are infeasible if and only if
1) $b > 0$ and $\Gamma(v, \bar{\delta}) = 1$
or 2) $b \leq 0$ and $\Gamma(v, \bar{\delta}) = -1$
- c) $1/2 \bar{\delta}$ is infeasible and $-1/2 \bar{\delta}$ is feasible if and only if
1) $b > 0$ and $\Gamma(v, \bar{\delta}) = -1$
or 2) $b \leq 0$ and $\Gamma(v, \bar{\delta}) = 1$
- d) $1/2 \bar{\delta}$ and $-1/2 \bar{\delta}$ are both infeasible if and only if
 $b > 0$ and $\Gamma(v, \bar{\delta}) = 0$

We first note that there are 3 values of $\Gamma(v, \bar{\delta})$ for each of the 2^{n-1} parallel pairs $\bar{\delta}$ and that the feasible sets differ with the sign of b (that is $b \leq 0$ or $b > 0$). Thus an upper bound on the number of equivalence classes is $2(3^{2^{n-1}})$. However (b) and (c) above indicate that different elements of these $2(3^{2^{n-1}})$ potential classes are the same. In particular if $\Gamma(v, \bar{\delta}) = \pm 1$ then changes of the sign of Γ and b produce the same equivalence class. Thus, since there are $2^{2^{n-1}}$ ways of assigning $\Gamma(v, \bar{\delta}) = \pm 1$ for all $\bar{\delta}$ and 2 ways of assigning the sign of b , there is a totality of $2^{(2^{n-1} + 1)}$ of these potential classes of which 1/2 are the same i.e., the change of sign of Γ and b . Thus from the total set of potential classes we must subtract those that are redundant. This yields

$$2(3^{2^{n-1}}) - 2^{2^{n-1}} \text{ equivalence classes.}$$

It is obvious that these equivalence classes can be determined by just observing the values of Γ and b as outlined in (a) through (d) of the above proof.

Example 1: Let us consider the possible equivalence classes in a two dimensional space. There are four vertices of K which we denote U, X, Y, Z . We let $U = (-1/2, -1/2)$, $X = (1/2, 1/2)$, $Y = (-1/2, 1/2)$ and $Z = (1/2, -1/2)$ thus these correspond to the following 0-1 points before translation of axis. $U' = (0, 0)$, $X' = (1, 1)$, $Y' = (0, 1)$ and $Z' = (1, 0)$.

Let $\bar{\delta}^1 = (1, 1)$ and $\bar{\delta}^2 = (-1, 1)$ then U, X represent the vertices classifiable by $\Gamma(v, \bar{\delta}^1)$, and Y and Z the vertices classifiable by $\Gamma(v, \bar{\delta}^2)$. The number of possible equivalence classes is determined from Theorem 1. There are $2^{n-1} = 2$ pairs of vertices and thus $2 \cdot 3^2 \cdot 2^2 = 14$ possible equivalence classes. These are as follows

	b	$\Gamma(v, \bar{\delta}^1)$	$\Gamma(v, \bar{\delta}^2)$	feasible vertices
1)	≤ 0	0	0	U, X, Y, Z
2)	≤ 0	0	-1	U, X, Y
3)	≤ 0	0	1	U, X, Z
4)	≤ 0	-1	0	X, Z, Y
5)	≤ 0	1	0	U, Z, Y
6)	$\begin{cases} \leq 0 \\ > 0 \end{cases}$	$\begin{cases} -1 \\ 1 \end{cases}$	$\begin{cases} -1 \\ 1 \end{cases}$	$\{X, Y\}$
7)	$\begin{cases} \leq 0 \\ > 0 \end{cases}$	$\begin{cases} -1 \\ 1 \end{cases}$	$\begin{cases} 1 \\ 1 \end{cases}$	$\{X, Z\}$
8)	$\begin{cases} \leq 0 \\ > 0 \end{cases}$	$\begin{cases} 1 \\ -1 \end{cases}$	$\begin{cases} -1 \\ 1 \end{cases}$	$\{U, Y\}$
9)	$\begin{cases} \leq 0 \\ > 0 \end{cases}$	$\begin{cases} 1 \\ -1 \end{cases}$	$\begin{cases} 1 \\ -1 \end{cases}$	$\{U, Z\}$
10)	> 0	0	0	none
11)	> 0	0	-1	Z
12)	> 0	0	1	Y
13)	> 0	-1	0	U
14)	> 0	1	0	X

One should note that the number of equivalence classes grows quite rapidly as shown below

n	2	3	4	5
Equivalence classes	14	146	12866	86,027,906

While we have been able to classify constraints into equivalence classes by Theorem 1, for any given constraint we must calculate 2^{n-1} functions $\Gamma(x, \bar{\delta})$ to determine this classification. Although this is not equivalent to enumerating all of the vertices of the hypercube, it is equivalent to enumerating half of the vertices. In the next section we investigate a partial ordering that exists on the extended facets of C and from this derive an efficient search algorithm for determining the appropriate equivalence class. In addition this partial ordering suggests a simple method of deriving other constraints in the equivalence class that are "better".

3. Calculations of $\Gamma(\cdot, \cdot)$

If we observe that the hypercube may be rotated and the axes relabeled we can assume without loss of generality that $ax \geq b$ has the property $0 \leq a_1 \leq a_2 \leq \dots \leq a_n$. Since the $\bar{\delta}$ for paired hyperplanes has the property $\sum \delta_i \geq 0$ we conclude that at most $\lceil \frac{n}{2} \rceil$ of the δ_i can be negative ($\lceil x \rceil$ is the greatest integer $\leq x$). We can thus describe $\bar{\delta}$ in terms of the negative elements of $\bar{\delta}$.

Let $k = \lceil n/2 \rceil$, $p = (p_1, \dots, p_k)$ be a k -dimensional vector, and $k^-(\bar{\delta})$ be the number of negative elements of $\bar{\delta}$. With every $\bar{\delta}$ we shall associate a unique vector $p(\bar{\delta})$ that has the following properties:

- a) $0 \leq p_1 \leq p_2 \leq \dots \leq p_k$
 - b) if $p_i \neq 0$, then $p_i < p_{i+1}$
 - c) $p_1 = p_2 = \dots = p_{k-k^-(\bar{\delta})} = 0$.
 - d) $\delta_i = -1$ if and only if $p_j = i$ for some j .
- (3.1)

Property (d) establishes the relationship for identifying $\bar{\delta}$ given $p(\bar{\delta})$, i.e., $p(\bar{\delta})$ is a list of the indices of the negative elements of $\bar{\delta}$. The uniqueness of $p(\bar{\delta})$ is established by properties of (a), (b) and (c). For notational convenience, we shall use the notation p for $p(\bar{\delta})$.

Example 2: For $n = 5$ $k = 2$ some examples of associated p are

$\bar{\delta} = (1, 1, 1, 1, 1)$	$p = (0, 0)$
$\bar{\delta} = (-1, 1, 1, 1, 1)$	$p = (0, 1)$
$\bar{\delta} = (1, -1, 1, -1, 1)$	$p = (2, 4)$

The p values introduce a vector partial ordering where $p \geq p'$ if and only if $p_i \geq p'_i$ for all i . If strict inequality must hold in at least one component we write $p > p'$. This partial ordering is important with respect to the indicator function $\Gamma(\cdot, \cdot)$; thus, in order to maintain notational consistency, we define a function $H(p; v)$ as follows:

Definition: $H(p; v) = \Gamma(v, \bar{\delta})$ where p is the vector associated with $\bar{\delta}$.

Definition: Two vectors $p \leq p^*$ are adjacent if $p_j = p_j^*$ for $j \neq i$ and $p_i^* = p_i + 1$.

We can now establish an important monotonicity relation on the function H .

Theorem 2: If $p \leq p'$ then $H(p; v) \geq H(p'; v)$ for $b > 0$ and $H(p; v) \leq H(p'; v)$ for $b \leq 0$.

Proof: Consider p and p^* such that $p \leq p^*$ and p^* is adjacent to p . Let

$p_i^* = p_i + 1$. The condition $p \leq p^*$ and property (a) of (3.1)

$p_j \neq p_i + 1$ for any j since otherwise, the number of non-zero values p^* would be one less than that of p by property (b) of (3.1). We also note that $H(p; v) \geq H(p^*; v)$ if and only if

$$\sum_{t=1}^n \delta_t(p) v_t \geq \sum_{t=1}^n \delta_t(p^*) v_t. \text{ Let } p_i = q. \text{ Then if } q = 0, \text{ we have}$$

$$\bar{\lambda} \sum_{t=1}^n \delta_t(p^*) a_t = -\bar{\lambda} a_1 + \bar{\lambda} \sum_{t=2}^n \delta_t(p) a_t$$

and

$$\bar{\lambda} \sum_{t=1}^n \delta_t(p) a_t = \bar{\lambda} a_1 + \bar{\lambda} \sum_{t=2}^n \delta_t(p) a_t.$$

Since $a_1 \geq 0$, then $H(p; v) \geq H(p^*; v)$ if $\bar{\lambda} > 0$ i.e., $b > 0$ and $H(p; v) \leq H(p^*; v)$ if $\bar{\lambda} < 0$ i.e., $b < 0$.

Now if $q \neq 0$ then

$$\bar{\lambda} \sum_{t=1}^n \delta_t(p^*) a_t = \bar{\lambda} \sum_{t \neq q, q+1} \delta_t(p) a_t - \bar{\lambda} (a_{q+1} - a_q)$$

and

$$\bar{\lambda} \sum_{t=1}^n \delta_t(p) a_t = \bar{\lambda} \sum_{t \neq q, q+1} \delta_t(p) a_t + \bar{\lambda} (a_{q+1} - a_q)$$

Since $a_{q+1} - a_q \geq 0$ by assumption then $H(p; v) \geq H(p^*; v)$ if $\bar{\lambda} > 0$
i.e. $b > 0$ and $H(p; v) \leq H(p^*; v)$ if $\bar{\lambda} < 0$ i.e. $b \leq 0$.

We now need to show that for any $p \leq p'$, p' can be reached by a path through adjacent vectors. Let $p_k = q$ and $p_k^i = q + r$. Then let $p^0 = p$ and $p^0 \leq p^1 \leq \dots \leq p^r$ where $p_k^i = p_k^{i-1} + 1$ for $i = 1, \dots, r$ and $p_j^i = p_j^{i-1}$ for $j \neq k$. This establishes monotonicity of H on p^0 through p^r by the above proof on adjacent vectors p^i and p^{i+1} . Also, $p^r \leq p^i$. We now take index $k-1$ and perform the same operation and continue with index $k-2, k-3, \dots, 1$ and thus have $p \leq p^0 \leq p^1 \leq \dots \leq p^m = p'$ where p^{i-1} and p^i differ by only one element. Thus, the monotonicity property is preserved and the theorem is proved.

Theorem 2 provides the basis for calculating the $\Gamma(v, \bar{\delta})$ by a search algorithm. Note that by the theorem one has only to calculate the vectors p where the function H changes value. Thus, if $H(p) = 1$ and $H(p^*) = 0$ with p and p^* adjacent and $b > 0$, we have $H(p^i) = 1$ for $p^i \leq p$ and $H(p^i) = 0$ or -1 for $p^i \geq p^*$. Thus, if we describe the set of p where H changes values we completely describe the function Γ and thus the equivalence class in which (a, b) exists. Towards this end, we define the following sets that characterize these points and thus, the equivalence class.

If $b \leq 0$ then

$$P^+ = \{ p \mid H(p;v) = 1 \text{ and } H(p';v) = 0 \text{ or } -1, p' \leq p \}$$

$$P^{0+} = \{ p \mid H(p;v) = 0 \text{ or } -1 \text{ and } H(p';v) = +1 \text{ for } p' \geq p \}$$

$$P^{0-} = \{ p \mid H(p;v) = 0 \text{ or } +1 \text{ and } H(p';v) = -1 \text{ for } p' \leq p \}$$

and $P^- = \{ p \mid H(p;v) = -1 \text{ and } H(p';v) = 0 \text{ or } 1, p' \geq p \}$

and if $b > 0$ then

$$P^+ = \{ p \mid H(p;v) = 1 \text{ and } H(p';v) = 0 \text{ or } 1, p' \geq p \}$$

$$P^{0+} = \{ p \mid H(p;v) = 0 \text{ or } -1 \text{ and } H(p';v) = +1 p' \leq p \}$$

$$P^{0-} = \{ p \mid H(p;v) = 0 \text{ or } +1 \text{ and } H(p';v) = -1 p' \geq p \}$$

and $P^- = \{ p \mid H(p;v) = -1 \text{ and } H(p';v) = 0 \text{ or } 1 p' \leq p \}$

From Lemma 2, the sets P^+ and P^- provide characterization of the feasible points as follows:

a) $1/2\bar{\delta}$ and $-1/2\bar{\delta}$ are both feasible if and only if $b \leq 0$ and

for some $p' \in P^-$ and some $p'' \in P^+$, $p' \leq p(\bar{\delta}) \leq p''$. (3.2a)

b) $1/2\bar{\delta}$ is feasible and $-1/2\bar{\delta}$ is infeasible if and only if either

1) $b > 0$ and for some $p' \in P^+$ $p(\bar{\delta}) \leq p'$ (3.2b)

or

2) $b \leq 0$ and for some $p' \in P^-$ $p(\bar{\delta}) \leq p'$.

c) $1/2\bar{\delta}$ is infeasible and $-1/2\bar{\delta}$ is feasible if and only if either

1) $b > 0$ and for some $p' \in P^-$ $p(\bar{\delta}) \geq p'$ (3.2c)

or

2) $b \leq 0$ and for some $p' \in P^+$ $p(\bar{\delta}) \geq p'$.

d) $1/2\bar{\delta}$ and $-1/2\bar{\delta}$ are both infeasible if and only if $b > 0$ and for some

$$p' \in P^+ \text{ and } p'' \in P^-$$

$$p' \leq p(\bar{\delta}) \leq p'' \quad (3.2d)$$

The sets P^{0+} and P^{0-} are used in the algorithm of the next section to determine P^+ and P^- and will be used to generate linear constraints describing the appropriate equivalence class in Section 5.

Example 3: Consider the constraint

$$2x_1 + 4x_2 + 5x_3 + 7x_4 + 8x_5 \geq -4$$

There are 16 possible vectors p . The pole is $v = -5/16(2, 4, 5, 7, 8)$

and the values of H are as follows:

p	$\delta(p)v$	$H(p, v)$	p	$\delta(p)v$	$H(p, v)$
00	$-\frac{65}{8}$	-1	14	$-\frac{20}{8}$	0
01	$-\frac{55}{8}$	-1	15	$-\frac{15}{8}$	0
02	$-\frac{45}{8}$	-1	23	$-\frac{20}{8}$	0
03	$-\frac{40}{8}$	-1	24	$-\frac{10}{8}$	0
04	$-\frac{30}{8}$	-1	25	$-\frac{5}{8}$	0
05	$-\frac{25}{8}$	-1	34	$-\frac{5}{8}$	0
12	$-\frac{35}{8}$	-1	35	0	0
13	$-\frac{30}{8}$	-1	45	$\frac{10}{8}$	0

Now we observe that $H(0, 5; v) = -1$ implies $H(0, t; v) = -1$ for

$t = 0, 1, \dots, 5$ and $H(1, 5; v) = 0$ implies $H(j, 5; v) = 0$, or 1 for

$j = 1, 2, 3, 4$, and since $H(4, 5; v) = 0$ then these are all zero. Since

$H(1, 4; v) = 0$ and $H(4, 5; v) = 0$ then $H(1, 5; v) = 0$ and we can classify

all of the values by $H(0,5) = -1$, $H(1,3) = -1$, $H(1,4) = 0$ and $H(2,3) = 0$.

Alternatively, we have $P^- = \{(0,5), (1,3)\}$, $P^{0-} = \{(1,4), (2,3)\}$, $P^{0+} =$
 $P^+ = \emptyset$.

In the next section we describe an algorithm for finding the sets
 P .

4. An Algorithm

In this section we present a method for determining the sets P^+ , P^{0+} , P^{0-} and P^- . The method determines elements in one set and then insures that all adjacent elements are dominated (in a partial ordered sense) by the elements of one of the sets. In the algorithm, we shall be interested in those adjacent vectors that are either greater than or less than a particular vector. Towards this end we use the following notation. The vectors adjacent to p' will be denoted as $p_1^i(j)$ $j = 1, \dots, n_1^i$ and $p_2^i(j)$ $j = 1, \dots, n_2^i$ where $p_1^i(j) \leq p'$ for all j and $p_2^i(j) \geq p'$ for all j . Thus, the $p_1^i(j)$ are the adjacent vectors to p' that are less than p' in partial ordering and $p_2^i(j)$ are the adjacent vectors that are greater than p' .

In addition, since the monotonicity of $H(p;v)$ is reversed for reversal of the sign of b the algorithm is expressed for $b > 0$ with changes for $b \leq 0$ designated in brackets. Finally, since we are constructing sets, we use $+$ and $-$ to denote the addition and subtraction of elements from sets.

Step 0: $P^+ = P^{0+} = P^- = \emptyset$. $T^+ = T^{0+} = T^{0-} = T^- = \emptyset$.

Step 1: Initial Search: Search an ordered chain from $p = (000, \dots, 0)$ to $p = (p_1, \dots, n-1, n)$, where $p_1 = n-k-1$ if n is odd and $p_1 = 1$ if n is even. Let p^* be the maximum element such that $H(p^*, v) = 1$ [$H(p^*, v) = -1$] and p^{**} be the minimum element such that $H(p^{**}, v) = -1$ [$H(p^{**}, v) = +1$]. Let $T^+ = \{p^*\}$ [$T^- = \{p^*\}$] and $T^- = \{p^{**}\}$ [$T^+ = \{p^{**}\}$].

Remark: It is possible that p^* and p^{**} may not exist. In this case, the appropriate set is left null. While any ordered chain may be used, a

convenient one is $(00, \dots, 0, j)$ $j = 0, 1, \dots, n$; $(000, \dots, 0, j, n)$ $j = 1, \dots, n-1$; $(000, \dots, 0, j, n-1, n)$ $j = 1, \dots, n-2$, etc.

Step 2: If $T^+ = \emptyset$ go to Step 3; otherwise, select any $p' \in T^+$ and $T^+ = T^+ - \{p'\}$.

2a: If $p' \leq p$ [$p' \geq p$] for any $p \in P^+$, go to Step 2; otherwise, $j = 1$ go to 2b.

2b: If $H(p_2^i(j), v) = 1$ [$H(p_1^i(j); v) = 1$], let $p' = p_2^i(j)$ [$p' = p_1^i(j)$] go to 2a; otherwise, $j = j + 1$ and go to 2c.

2c: If $j > n_2^i$ [$j > n_1^i$] go to 2d; otherwise, go to 2b.

2d: $P^+ = P^+ + \{p'\}$, $T^{0+} = T^{0+} + \{p_2^i(1)\} + \dots + \{p_2^i(n_2^i)\}$ [$T^{0+} = T^{0+} + \{p_1^i(1)\} + \dots + \{p_1^i(n_1^i)\}$]. Go to 2.

Step 3: If $T^{0+} = \emptyset$, go to step 4.

Otherwise, select any $p' \in T^{0+}$ and $T^{0+} = T^{0+} - \{p'\}$.

3a: If $p' \geq p$ [$p' \leq p$] for any $p \in P^{0+}$, go to step 3. Otherwise, $j = 1$, go to 3b.

3b: If $H(p_1^i(j); v) = 0$ or -1 [$H(p_2^i(j); v) = 0$ or -1] let $p' = p_1^i(j)$ [$p' = p_2^i(j)$] go to 3a; Otherwise, $j = j + 1$ and go to 3c.

3c: If $j > n_1^i$ [$j > n_2^i$] go to 3d; Otherwise, go to 3b.

3d: $P^{0+} = P^{0+} + \{p'\}$, $T^+ = T^+ + \{p_1^i(1)\} + \dots + \{p_1^i(n_1^i)\}$ [$T^+ = T^+ + \{p_2^i(1)\} + \dots + \{p_2^i(n_2^i)\}$]. Go to 2.

Remark: Upon entering step 4 we have determined the sets P^+ and P^{0+} .

Step 4 and 5 are analogous to Step 2 and 3 and determine P^- and P^{0-} .

Step 4: If $T^- = \emptyset$, go to Step 5; Otherwise, select any $p' \in T^-$ and $T^- = T^- - \{p'\}$.

- 4a: If $p' \geq p$ [$p' \leq p$] for any $p \in P^-$, go to Step 4; Otherwise,
 $j = 1$, go to 4b.
- 4b: If $H(p'_1(j);v) = -1$ [$H(p'_2(j);v) = -1$] let $p' = p'_1(j)$ [$p' = p'_2(j)$]
go to 4a; Otherwise, $j = j + 1$ and go to 4c.
- 4c: If $j > n'_1$ [$j > n'_2$] go to 4d; Otherwise, go to 4b.
- 4d: $P^- = P^- + \{p'\}$, $T^{0-} = T^{0-} + \{p'_1(1)\} + \dots + \{p'_1(n'_1)\}$ [$T^{0-} =$
 $T^{0-} + \{p'_2(1)\} + \dots + \{p'_2(n'_2)\}$]. Go to 4.
- Step 5: If $T^{0-} = \emptyset$ stop; Otherwise, select any $p' \in T^{0-}$ and let $T^{0-} =$
 $T^{0-} - \{p'\}$.
- 5a: If $p' \leq p$ [$p' \geq p$] for any $p \in P^{0-}$, go to Step 5: Otherwise,
 $j = 1$. Go to 5b.
- 5b: If $H(p'_2(j);v) = 0$ or $+1$ [$H(p'_1(j);v) = 0$ or $+1$], let $p' = p'_2(j)$
 $[p' = p'_1(j)]$. Go to 5a; Otherwise, $j = j + 1$ and go to 5c.
- 5c: If $j > n'_2$ [$j > n'_1$], go to 2d; Otherwise go to 5b.
- 5d: $P^{0-} = P^{0-} + \{p'\}$, $T^- = T^- + \{p'_2(j)\} + \dots + \{p'_2(n'_2)\}$ [$T^- =$
 $T^- + \{p'_1(1)\} + \dots + \{p'_1(n'_1)\}$]. Go to 4.

In the next section, we discuss a possible application of equivalence classes to determine a "best" constraint for the equivalence classes.

5. Generating Constraints from Equivalence Classes

Let us now suppose that we have an equivalence class as represented by the sets P^+ and P^- . A question then arises as to what constraint(s) should we use to represent this equivalence class and how can they be obtained. We represent our new constraint as

$$\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n \geq \beta_0.$$

Since $H(p;v)$ is a function after both rotation and permutation of the axis have been eliminated we have the additional conditions

$$0 \leq \beta_1 \leq \beta_2 \leq \dots \leq \beta_n.$$

In addition, we also know that the class is determined by the sets P^+ and P^- of the function $H(p;v)$. We therefore, construct a set of linear constraints using these sets and P^{0+} and P^{0-} .

From the sets P^+ , P^{0+} , P^{0-} and P^- and (3.2) we have the following conditions:

If $b \leq 0$ then

$$-1/2\delta(p)\beta \leq \beta_0 - 1/2 \quad \text{for } p \in P^- \quad (5.1a)$$

$$1/2\delta(p)\beta \leq \beta_0 - 1/2 \quad \text{for } p \in P^+ \quad (5.1b)$$

$$-1/2\delta(p)\beta \geq \beta_0 \quad \text{for } p \in P^{0-} \quad (5.1c)$$

$$1/2\delta(p)\beta \geq \beta_0 \quad \text{for } p \in P^{0+} \quad (5.1d)$$

and if $b > 0$ then

$$-1/2\delta(p)\beta \geq \beta_0 \quad \text{for } p \in P^- \quad (5.1e)$$

$$1/2\delta(p)\beta \geq \beta_0 \quad \text{for } p \in P^+ \quad (5.1f)$$

$$-1/2\delta(p)\beta \leq \beta_0 - 1/2 \quad \text{for } p \in P^{0-} \quad (5.1g)$$

$$1/2\delta(p)\beta \leq \beta_0 - 1/2 \quad \text{for } p \in P^{0+} \quad (5.1h)$$

These conditions come directly from Lemma 2. We note that P^+ and P^{0-} are those points that are adjacent in the sense that if $b \leq 0$, $-1/2\delta(p)$ is feasible for $p \in P^{0-}$ while $-1/2\delta(p)$ is infeasible for $p \in P^+$. This fact is reflected in statements (5.1c) and (5.1a). We use $\beta_0 = 1/2$ for infeasibility since this translates back to $\beta'_0 = 1$ (i.e., integer r.h.s.) indicating infeasibility in the x' (zero-one) space. The linear constraints (5.1) have the property that all $\beta = (\beta_0, \beta_1, \dots, \beta_n)$ satisfying them belong to the same equivalence class and any β that does not satisfy these constraints does not belong to the same equivalence class as the constraint $(ax \geq b)$ from which the sets P^+, P^- were derived.

The criteria of choosing a "best" constraint from the set of solutions to the linear constraints will of course depend on the definition of the term "best". We propose that one meaningful definition is that β provides a best constraint if the number of binding (equality) constraints for the defining linear inequalities (5.1) is as large as possible. From linear programming theory we know that these conditions corresponds to generating basic feasible solutions for the constraints (5.1), where the number of non-basic slacks is as large as possible. This immediately implies that there may be several "best" constraints since there may be several such basic solutions. However, they all possess the property that a slight perturbation of a coefficient either makes one of the defining equalities non-binding or makes β no longer satisfying the constraint set. It should also be noted that there are $n+1$ variables in (5.1). While this may seem disconcerting since we need only n points to determine a hyperplane in n space, it is a result of our definition of an infeasible point as being at least 1 unit

length away from the constraint in terms of the right-hand side. One may envision this process as choosing n points to determine the hyperplane and then one additional point so that a displacement of the hyperplane by one unit (in the x' space) will pass through an infeasible point, see (5.1a,b,g or h).

Remark:

We note that the constraint (5.1) can be translated back to the x' space since $x_i = -1/2$ implies $x_i' = 0$ and $x_i = 1/2$ implies $x_i' = 1$. Since this reduces the number of non-zero coefficients in the constraint set produced, we propose that this should be used.

Example 4:

Let us now consider the constraint of Example 3. In that example we had $P^{0+} = P^+ = \emptyset$, and $P^- = [(0,5), (1,3)]$ and $P^{0-} = [(1,4), (2,3)]$.

The constraints from (5.1) are thus,

$$\begin{aligned} \beta_5 &\leq \beta_0' - 1 \\ \beta_1 + \beta_3 &\leq \beta_0' - 1 \\ \beta_1 + \beta_4 &\geq \beta_0' \\ \beta_2 + \beta_3 &\geq \beta_0' \\ 0 \leq \beta_1 \leq \beta_2 \leq \beta_3 \leq \beta_4 \leq \beta_5 \end{aligned} \tag{5.2}$$

The first constraint comes from the fact that $p = (0,5)$ implies $\bar{\delta} = (1,1,1, 1,-1)$ and since $-1/2\bar{\delta}$ is infeasible, then $x_5' = 1, x_1' = x_2' = x_3' = x_4' = 0$ is infeasible. The other constraints are derived similarly.

We also note that we can change variables and eliminate the constraints $0 \leq \beta_1 \leq \beta_2 \leq \beta_3 \leq \beta_4 \leq \beta_5$. Let $y_1 = \beta_1$ and $y_i = \beta_i - \beta_{i-1}$ for $i \geq 2$ then $y_i \geq 0$ implies $\beta_i \geq \beta_{i-1}$ thus, (5.2) reduces to finding a

solution to

$$\begin{aligned} y_1 + y_2 + y_3 + y_4 + y_5 &\leq \beta'_0 - 1 \\ 2y_1 + y_2 + y_3 &\leq \beta'_0 - 1 \\ 2y_1 + y_2 + y_3 + y_4 &\geq \beta'_0 \\ 2y_1 + 2y_2 + y_3 &\geq \beta'_0 \\ y_i &\geq 0 \end{aligned}$$

Solving this set of constraints we find that $y_1 = 1, y_2 = 1, y_3 = 0, y_4 = 1, y_5 = 0, \beta'_0 = 4$ is a basic feasible solution (and the only basic feasible solution). This yields the constraint $x'_1 + 2x'_2 + 2x'_3 + 3x'_4 + 3x'_5 \geq 4$. It is easy to check that this is from the same equivalence class as the original constraint.

Remarks: 1) The equations (5.1) indicate that the flexibility of "moving" constraints and maintaining the same feasible vertices of the hypercube is dependent not only on keeping certain vertices feasible i.e., equations (5.1c), (5.1d), (5.1e) and (5.1f) but also on keeping certain vertices infeasible, i.e., equations (5.1a), (5.1b), (5.1g) and (5.1h).

2) One should not interpret the constraint generation of this section as being the only or even the main application of theory of constraint equivalence classes. In [5], the concept of equivalence classes is directly applied to a special class of 0-1 programming problems resulting in not only the optimal solution but direct sensitivity analysis on the changes in the objective function.

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